Univariate Linear Regression 单变量线性回归

1. Form and Assumption 形式 & 假设

observation $\{(x_i, y_i) \mid i = 1, 2, \cdots, n\}$

① Form of univariate linear regression:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

covariance x_i is nonrandom

response y_i

intercept β_0 is unkown parameters

slope β_1 is unkown parameters

error ε_i is random variable

误差 (y_i 不能被 x_i 解释的部分)

② Assumption of ε_i 只要不相关，不需要独立

Gauss-Markov condition : for (unknown) constant σ^2

$$E(\varepsilon_i) = 0$$

$$\operatorname{cov}(\varepsilon_i, \varepsilon_j) = \begin{cases} 0 & i \neq j \\ \sigma^2 & i = j \end{cases}$$

③ Based on assumption.

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

$$E(y_i) = \beta_0 + \beta_1 x_i$$

$$\operatorname{cov}(y_i, y_j) = \begin{cases} 0 & i \neq j \\ \sigma^2 & i = j \end{cases}$$

2. Estimation of the regression parameters 估计 $\beta_0, \beta_1, \sigma^2$

① Method 1 : Ordinary least squares estimation (OLSE) 普通最小二乘法

Intuition Loss function

$$Q(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i - E(y_i))^2$$
$$= \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

Solution derivative of quadratic loss function

$$\left.\frac{\partial Q}{\partial \beta_0}\right|_{\beta_0 = \hat{\beta}_0} = -2 \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

二阶导

$$\left.\frac{\partial Q}{\partial \beta_1}\right|_{\beta_1 = \hat{\beta}_1} = -2 \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) x_i = 0$$

$$\Rightarrow \quad \hat{\beta}_0 = \bar{y} - \bar{x} \hat{\beta}_1$$
$$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \frac{cov(x, y)}{var(x)} = \frac{L_{xy}}{L_{xx}}$$

Proof:

$\hat{\beta}_0$: $\sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0 \Rightarrow n\hat{\beta}_0 = \sum y_i - \hat{\beta}_1 \sum x_i \Rightarrow \hat{\beta}_0 = \bar{y} - \bar{x}\hat{\beta}_1$

$\hat{\beta}_1$: $\sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) x_i = 0 \Rightarrow \sum (y_i - \bar{y} + \bar{x}\hat{\beta}_1 - \hat{\beta}_1 x_i) x_i = 0$

$$LHS = \sum x_i y_i - \bar{y} \sum x_i + \bar{x}\hat{\beta}_1 \sum x_i - \hat{\beta}_1 \sum x_i^2 = 0$$

$$\therefore \quad \hat{\beta}_1 \sum x_i (x_i - \bar{x}) = \sum x_i (y_i - \bar{y})$$

$$\hat{\beta}_1 \sum (x_i - \bar{x} + \bar{x})(x_i - \bar{x}) = \sum (x_i - \bar{x} + \bar{x})(y_i - \bar{y})$$

it is obvious that $\sum (x_i - \bar{x}) = \sum x_i - n\bar{x} = 0$ $\quad \sum (y_i - \bar{y}) = 0$

$$\therefore \quad \hat{\beta}_1 \sum (x_i - \bar{x})^2 = \sum (x_i - \bar{x})(y_i - \bar{y})$$

$$\Rightarrow \hat{\beta}_1 = \frac{cov(x, y)}{var(x)} = \frac{L_{xy}}{L_{xx}}$$

Remark Sum of Squares of the Residuals (Errors) 残差平方和

$$e_i = y_i - \hat{y}_i$$

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

$$SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

Property

$$\sum_{i=1}^{n} e_i = 0$$

$$\sum_{i=1}^{n} x_i e_i = 0$$

Proof:

$$\begin{aligned}
\Sigma e_i &= \Sigma (y_i - \hat{y}_i) = \Sigma (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) \\
&= \Sigma y_i - \Sigma (\bar{y} - \bar{x}\hat{\beta}_1) - \Sigma \hat{\beta}_1 x_i \\
&= (\Sigma y_i - n\bar{y}) + n\bar{x}\hat{\beta}_1 - \hat{\beta}_1 \Sigma x_i \\
&= \hat{\beta}_1 (n\bar{x} - \Sigma x_i) \\
&= 0 \quad \square
\end{aligned}$$

$$\begin{aligned}
\Sigma x_i e_i &= \Sigma x_i (y_i - \hat{y}_i) = \Sigma x_i y_i - \Sigma x_i (\hat{\beta}_0 - \hat{\beta}_1 x_i) \\
&= \Sigma x_i y_i - \Sigma x_i (\bar{y} - \bar{x}\hat{\beta}_1 - \hat{\beta}_1 x_i) \\
&= \Sigma x_i y_i - \bar{y}\Sigma x_i + \bar{x}\hat{\beta}_1 \Sigma x_i + \hat{\beta}_1 \Sigma x_i^2 \\
&= \Sigma x_i (y_i - \bar{y}) + \hat{\beta}_1 \Sigma x_i (\bar{x} - x_i)
\end{aligned}$$

$$\because \hat{\beta}_1 \Sigma x_i (x_i - \bar{x}) = \Sigma x_i (y_i - \bar{y})$$

$$\begin{aligned}
&= \Sigma x_i (y_i - \bar{y}) - \Sigma x_i (y_i - \bar{y}) \\
&= 0 \quad \square
\end{aligned}$$

Remark: $\hat{\beta}_1, \hat{\beta}_2$ are random variables

while β_1, β_2 are (unknown) constants

② Method 2 : Maximum Likelihood estimation (MLE) 极大似然估计

Add Assumption

y_i are independent

i.i.d. $\varepsilon_i \sim N(0, \sigma^2)$

therefore, $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$

Intuition max Likelihood function (or Log-Likelihood function)

$$f_i(y_i) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left\{-\frac{1}{2\sigma^2} \cdot [y_i - (\beta_0 + \beta_1 x_i)]^2\right\}$$

$$L(\beta_0, \beta_1, \sigma^2) = \prod_{i=1}^{n} f_i(y_i \mid \beta_0, \beta_1, \sigma^2)$$

$$\log(L) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \cdot \sum_{i=1}^{n} [y_i - (\beta_0 + \beta_1 x_i)]^2$$

Solution

$\hat{\beta}_0$, $\hat{\beta}_1$ are the same

$$\left.\frac{\partial \log(L)}{\partial \sigma^2}\right|_{\sigma^2 = \hat{\sigma}^2} = -\frac{n}{2} \cdot \frac{1}{2\pi\hat{\sigma}^2} + \frac{1}{4\hat{\sigma}^4} \cdot \sum_{i=1}^{n} [y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)]^2$$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

二阶导

Remark $\hat{\sigma}^2 = \frac{1}{n}\sum(y_i - \hat{y}_i)^2$ is biasedness

$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

$$= \frac{1}{n-2}\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$ is unbiasedness

Remark: assuming $\varepsilon_i \sim N(0, \sigma^2)$, $\varepsilon_j \sim N(0, \sigma^2)$ and $\mathrm{cov}(\varepsilon_i, \varepsilon_j) = 0$ does **not** ensure $\varepsilon_i \perp \varepsilon_j$.

Property $\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i-\hat{y}_i)^2$

$= \frac{1}{n-2}\sum_{i=1}^{n}(y_i-\hat{\beta}_0-\hat{\beta}_1 x_i)^2$ is unbiasedness

Proof 见后

3. Properties of Ordinary least squares estimation 最小二乘性质

☆ 探究参数 β_0, β_1 的估计 $\hat{\beta}_0, \hat{\beta}_1$ 的性质

① Linearity

$\hat{\beta}_0, \hat{\beta}_1$ are linear functions of y_i

$$\hat{\beta}_1 = \sum_{i=1}^{n}\frac{x_i-\bar{x}}{\sum_{j=1}^{n}(x_j-\bar{x})^2}\cdot y_i$$

$$\hat{\beta}_0 = \bar{y}-\hat{\beta}_1\bar{x} = \sum_{i=1}^{n}\left\{\frac{1}{n}-\frac{\bar{x}(x_i-\bar{x})}{\sum_{j=1}^{n}(x_j-\bar{x})^2}\right\}\cdot y_i$$

Proof: $\hat{\beta}_1 = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{L_{xx}}$

$= \frac{1}{L_{xx}}\cdot\sum(x_i-\bar{x})y_i - \frac{1}{L_{xx}}\cdot\sum(x_i-\bar{x})\bar{y}$

$= \frac{1}{L_{xx}}\sum(x_i-\bar{x})\cdot y_i - 0$ □

$\hat{\beta}_0 = \bar{y}-\bar{x}\cdot\hat{\beta}_1 = \frac{1}{n}\sum y_i - \frac{1}{L_{xx}}\cdot\sum\bar{x}(x_i-\bar{x})\cdot y_i$

$= \sum_{i=1}^{n}\left\{\frac{1}{n}-\frac{\bar{x}(x_i-\bar{x})}{\sum_{j=1}^{n}(x_j-\bar{x})^2}\right\}\cdot y_i$ □

② Unbiasedness

$E(\hat{x})=x$ 这在实践中是有利的，表明，重复次数增多，估计值的平均会越来越靠近真实值

$$E(\hat{\beta}_1) = \beta_1$$

$$E(\hat{\beta}_0) = \beta_0$$

Proof:

$$E(\hat{\beta}_1) = \sum \frac{x_i - \bar{x}}{L_{xx}} \cdot E(y_i)$$
$$= \sum \frac{x_i - \bar{x}}{L_{xx}} \cdot (\beta_0 + \beta_1 x_i)$$
$$= \frac{\beta_0}{L_{xx}} \cdot \sum (x_i - \bar{x}) + \frac{\beta_1}{L_{xx}} \sum x_i (x_i - \bar{x})$$
$$= 0 + \frac{\beta_1}{L_{xx}} \cdot \sum (x_i - \bar{x} + \bar{x})(x_i - \bar{x})$$
$$= \frac{\beta_1}{L_{xx}} \cdot L_{xx} + 0$$
$$= \beta_1 \quad \square$$

$$E(\hat{\beta}_0) = E(\bar{y} - \bar{x}\hat{\beta}_1)$$
$$= \frac{1}{n} \sum E(y_i) - \bar{x} \cdot E(\hat{\beta}_1)$$
$$= \frac{1}{n} \cdot \sum (\beta_0 + \beta_1 x_i) - \bar{x} \cdot \beta_1$$
$$= \frac{1}{n} \cdot n\beta_0 + \beta_1 \cdot \frac{1}{n} \sum x_i - \beta_1 \cdot \bar{x}$$
$$= \beta_0 \quad \square$$

③ Variance and $cov(\hat{\beta}_1, \hat{\beta}_0)$

$$var(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$var(\hat{\beta}_0) = \left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right] \sigma^2$$

$$cov(\hat{\beta}_1, \hat{\beta}_0) = - \frac{\bar{x}}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \cdot \sigma^2$$

Proof: $var(\hat{\beta}_1) = var\left(\sum \frac{x_i-\bar{x}}{L_{xx}} \cdot y_i\right)$

$= \sum \left(\frac{x_i-\bar{x}}{L_{xx}}\right)^2 \cdot var(y_i)$

$= \frac{\sum (x_i-\bar{x})^2}{L_{xx}^2} \cdot \sigma^2$

$= \frac{\sigma^2}{L_{xx}}$ $\square$

$var(\hat{\beta}_0) = var(\bar{y} - \bar{x}\hat{\beta}_1)$

$= var(\bar{y}) + var(\bar{x}\hat{\beta}_1) + 0$

$= \left(\frac{1}{n^2} \sum var(y_i) + 0 + 0 + \cdots + 0\right) + \bar{x}^2 \cdot \frac{\sigma^2}{L_{xx}}$

$= \frac{1}{n} \cdot \sigma^2 + \bar{x}^2 \cdot \frac{\sigma^2}{L_{xx}}$

$= \left(\frac{1}{n} + \frac{\bar{x}^2}{L_{xx}}\right) \sigma^2$ $\square$

$cov(\hat{\beta}_1, \hat{\beta}_0) = E(\hat{\beta}_1 - E(\hat{\beta}_1))(\hat{\beta}_0 - E(\hat{\beta}_0))$

$= E(\hat{\beta}_1 - \beta_1)(\hat{\beta}_0 - \beta_0)$

$= E(\hat{\beta}_1 - \beta_1)(\bar{y} - \bar{x}\hat{\beta}_1 - \beta_0)$

$= E[(\hat{\beta}_1 - \beta_1)\bar{y}] - E[(\hat{\beta}_1 - \beta_1) \cdot \bar{x}\hat{\beta}_1] - E[(\hat{\beta}_1 - \beta_1)\beta_0]$

$\because E(\hat{\beta}_1 - \beta_1) = E(\hat{\beta}_1) - \beta_1 = 0$

$\bar{y} = \frac{1}{n}\sum(\beta_0 + \beta_1 x_i)$ $\beta_0, \beta_1, x_i \in const$

$= 0 - \bar{x} E(\hat{\beta}_1^2 - \beta_1\hat{\beta}_1) - 0$

$= -\bar{x} E(\hat{\beta}_1^2) + \bar{x}\beta_1 E(\hat{\beta}_1)$

$= -\bar{x}\left[var(\hat{\beta}_1) + (E(\hat{\beta}_1))^2\right] + \bar{x}\beta_1^2$

$= -\bar{x} \cdot \frac{\sigma^2}{L_{xx}} - \bar{x}\beta_1^2 + \bar{x}\beta_1^2$

$= -\frac{\bar{x}}{L_{xx}} \cdot \sigma^2$ $\square$

$$\left.\begin{aligned} \hat{\beta}_1 &= \sum_{i=1}^{n} \frac{x_i-\bar{x}}{\sum_{j=1}^{n}(x_j-\bar{x})^2} \cdot y_i \\ \hat{\beta}_0 &= \bar{y} - \hat{\beta}_1\bar{x} = \sum_{i=1}^{n}\left\{\frac{1}{n} - \frac{\bar{x}(x_i-\bar{x})}{\sum_{j=1}^{n}(x_j-\bar{x})^2}\right\} \cdot y_i \end{aligned}\right\} cov(\hat{\beta}_1, \hat{\beta}_0)$$

④ Distribution

Add Assumption

① $\varepsilon_i \sim N(0, \sigma^2)$

② $y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$, y_i independent

$\because$ Linearity

$$\hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)$$

$$\hat{\beta}_0 \sim N\left(\beta_0, \left[\frac{1}{n}+\frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right]\sigma^2\right)$$

⑤ MLE Assume $\varepsilon_i \sim N(0, \sigma^2)$

残差 $e_i = y_i - \hat{y}_i$ 与 $\hat{\beta}_1$, $\hat{\beta}_0$ 不相关（不代表独立）

$$cov(e_i, \hat{\beta}_1) = 0$$

$$cov(e_i, \hat{\beta}_0) = 0$$

Proof:

$$cov(e_i, \hat{\beta}_0) = cov(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i, \hat{\beta}_0)$$

$$= cov(y_i, \hat{\beta}_0) - var(\hat{\beta}_0) - x_i \cdot cov(\hat{\beta}_1, \hat{\beta}_0)$$

$\because$ $\hat{\beta}_0 = \sum\left(\frac{1}{n} - \frac{\bar{x}(x_i-\bar{x})}{L_{xx}}\right)\cdot y_i$ $\quad cov(y_i, y_j) = 0 \quad i \neq j$

$\therefore$ $= \left(\frac{1}{n} - \frac{\bar{x}(x_i-\bar{x})}{L_{xx}}\right)\cdot\sigma^2 - \left(\frac{1}{n}+\frac{\bar{x}^2}{L_{xx}}\right)\sigma^2 + x_i \cdot \frac{\bar{x}}{L_{xx}}\cdot\sigma^2$

$= 0 \quad \square$

$$cov(e_i, \hat{\beta}_1) = cov(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i, \hat{\beta}_1)$$

$$= cov(y_i, \hat{\beta}_1) - cov(\hat{\beta}_0, \hat{\beta}_1) - x_i \cdot var(\hat{\beta}_1)$$

$\because$ $\hat{\beta}_1 = \sum \frac{x_i - \bar{x}}{L_{xx}} y_i$ $\quad cov(y_i, y_j) = 0 \quad i \neq j$

$$= \frac{x_i - \bar{x}}{L_{xx}}\sigma^2 + \frac{\bar{x}}{L_{xx}}\sigma^2 - x_i \cdot \frac{\sigma^2}{L_{xx}}$$

$= 0 \quad \square$

⑥ MLE. $\hat{\sigma}^2$ 的无偏估计　　见 1.2

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$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i-\hat{y}_i)^2$$

$$= \frac{1}{n-2}\sum_{i=1}^{n}(y_i-\hat{\beta}_0-\hat{\beta}_1 x_i)^2 \quad \text{is} \quad \text{unbiasedness}$$

Rm. y_i, $\hat{\beta}_0$, $\hat{\beta}_1$ 是随机变量，β_0, β_1, x_i 为定常数

Proof $e_i = y_i - \hat{y}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i$

$$E(\Sigma e_i^2) = E(\Sigma(y_i-\hat{y}_i)^2) = \Sigma \underbrace{E(y_i-\hat{y}_i)^2}_{①}$$

$$① = [E\underbrace{(y_i-\hat{y}_i)}_{②}]^2 + \underbrace{var(y_i-\hat{y}_i)}_{③}$$

$$② = E(y_i) - E(\hat{y}_i) = \beta_0+\beta_1x_i - \beta_0 - \beta_1 x_i = 0$$

$$③ = var(y_i)+var(\hat{y}_i) - 2\,cov(y_i, \hat{y}_i)$$

$$= \sigma^2 + \underbrace{var(\hat{\beta}_0+\hat{\beta}_1 x_i)}_{④} - 2\underbrace{cov(y_i, \hat{\beta}_0+\hat{\beta}_1 x_i)}_{⑤}$$

$$④ = var(\hat{\beta}_0) + x_i^2 \cdot var(\hat{\beta}_1) + 2x_i\, cov(\hat{\beta}_0, \hat{\beta}_1)$$

$$= \left(\frac{1}{n}+\frac{\bar{x}^2}{L_{xx}}\right)\sigma^2 + x_i^2\cdot\frac{1}{L_{xx}}\sigma^2 - 2x_i\frac{\bar{x}}{L_{xx}}\sigma^2$$

$$= \frac{\sigma^2}{n} + \frac{\sigma^2}{L_{xx}}(\bar{x}^2+x_i^2-2\bar{x}x_i)$$

$$= \frac{\sigma^2}{n} + \frac{\sigma^2}{L_{xx}}(x_i-\bar{x})^2$$

$$⑤ = cov(y_i, \hat{\beta}_0) + x_i \cdot cov(y_i, \hat{\beta}_1)$$

$$cov(y_i, \hat{\beta}_0) = \left(\frac{1}{n} - \frac{\bar{x}(x_i-\bar{x})}{L_{xx}}\right)\cdot\sigma^2$$

$$cov(y_i, \hat{\beta}_1) = \frac{x_i-\bar{x}}{L_{xx}}\sigma^2$$

$$= \frac{\sigma^2}{n} - \frac{\bar{x}(x_i-\bar{x})}{L_{xx}}\sigma^2 + \frac{x_i(x_i-\bar{x})}{L_{xx}}\sigma^2$$

$$= \frac{\sigma^2}{n} + \frac{\sigma^2}{L_{xx}}(x_i-\bar{x})\cdot(x_i-\bar{x}) = \frac{\sigma^2}{n} + \frac{(x_i-\bar{x})^2}{L_{xx}}\sigma^2$$

$$\therefore ① = E(y_i-\hat{y}_i)^2$$

$$= \sigma^2 + \frac{\sigma^2}{n} + \frac{\sigma^2}{L_{xx}}(x_i-\bar{x})^2 - 2\left(\frac{\sigma^2}{n} + \frac{(x_i-\bar{x})^2}{L_{xx}}\sigma^2\right)$$

$$= \sigma^2\left(1+\frac{1}{n} + \frac{1}{L_{xx}}(x_i-\bar{x})^2 - \frac{2}{n} - \frac{2}{L_{xx}}\cdot(x_i-\bar{x})^2\right)$$

$$= \sigma^2\left(1-\frac{1}{n} - \frac{1}{L_{xx}}(x_i-\bar{x})^2\right)$$

$$\therefore E(\hat{\sigma}^2) = \frac{1}{n-2}\sum E(e_i^2)$$

$$= \frac{1}{n-2}\sum \sigma^2\left(1-\frac{1}{n}-\frac{(x_i-\bar{x})^2}{L_{xx}}\right)$$

$$= \frac{1}{n-2}\left(n-1-\sum\frac{(x_i-\bar{x})^2}{L_{xx}}\right)\sigma^2$$

$$= \frac{1}{n-2}(n-1-1)\sigma^2$$

$$= \sigma^2 \quad \square$$

⑦. 对于固定的 i， $\hat{y}_i$ 也是 $y_1,\cdots,y_n$ 的线性组合

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

$$\hat{y}_i \sim N\left(\beta_0+\beta_1 x_i,\ \left(\frac{1}{n}+\frac{(x_i-\bar{x})}{L_{xx}}\right)\sigma^2\right)$$

Remark: x_i 固定 (i 固定)

Proof:

① $E(\hat{y}_i) = E(\hat{\beta}_0) + E(\hat{\beta}_1)x_i = \beta_0 + \beta_1 x_i \quad \square$

② $Var(\hat{y}_i) = Var(\hat{\beta}_0 + \hat{\beta}_1 x_i)$

$$= Var(\hat{\beta}_0) + Var(\hat{\beta}_1 x_i) + 2\,cov(\hat{\beta}_0, \hat{\beta}_1 x_i)$$

$$= \left(\frac{1}{n}+\frac{\bar{x}^2}{L_{xx}}\right)\sigma^2 + x_i^2\cdot\frac{\sigma^2}{L_{xx}} - 2x_i\cdot\frac{\bar{x}}{L_{xx}}\sigma^2$$

$$= \left(\frac{1}{n}+\frac{(x_i-\bar{x})}{L_{xx}}\right)\sigma^2 \quad \square$$